$RO(S^1)$-GRADED TR–GROUPS OF $\mathbb{F}_p$, $\mathbb{Z}$ AND ℓ

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Abstract. We give an algorithm for calculating the $RO(S^1)$-graded TR–groups of $\mathbb{F}_p$, completing the calculation started in [9]. We also calculate the $RO(S^1)$-graded TR–groups of $\mathbb{Z}$ with mod p coefficients and of the Adams summand ℓ of connective complex K-theory with $V(1)$-coefficients. Some of these calculations are used in [2] and [1] to compute the algebraic K-theory of certain $\mathbb{Z}$-algebras.

1. Introduction

Higher algebraic K-theory associates to a ring or ring spectrum A a spectrum $K(A)$ and a sequence of abelian groups $K_i(A)$ which are the homotopy groups of this spectrum. Although higher algebraic K-theory was defined more than 30 years ago, computational progress has been slow. While the definition of algebraic K-theory is not inherently equivariant, the tools of equivariant stable homotopy theory have proven useful for K-theory computations via an approach referred to as trace methods [5]. The equivariant stable homotopy computations in this paper serve as input for this method. In particular they have been used in the computations of the relative algebraic K-theory groups $K_*(\mathbb{Z}[x]/(x^m),(x))$ and $K_*(\mathbb{Z}[x,y]/(xy),(x,y))$ up to extensions (see [2] and [1] respectively).

The idea behind the trace methods approach is to approximate algebraic K-theory by invariants of ring spectra which are more computable. The first approximation is topological Hochschild homology [7], $T(A)$. This is significantly easier to compute than algebraic K-theory and there is a trace map $K(A) \to T(A)$, called the topological Dennis trace. A refinement of topological Hochschild homology called topological cyclic homology, $\mathrm{TC}(A)$, serves as an even better approximation of algebraic K-theory. Indeed, there is a map $trc : K(A) \to \mathrm{TC}(A)$, called the cyclotomic trace [5] which is often close to an equivalence [13, 16, 8]. So in good cases the trace methods approach reduces the computation of algebraic K-theory, $K_q(A)$, to that of topological cyclic homology, $\mathrm{TC}_q(A)$.

Topological cyclic homology is defined by looking at fixed points of topological Hochschild homology. Let p be a prime. The circle S^1 acts on $T(A)$, and we define $\mathrm{TR}^n(A;p) = T(A)^{C_{p^{n-1}}}$ as the fixed point spectrum under the action of the cyclic group of order p^{n-1} considered as a subgroup of S^1. These spectra are connected by maps R, F, V and d [15], and a homotopy limit over R and F gives us the topological cyclic homology spectrum $\mathrm{TC}(A;p)$. So, to compute topological cyclic homology, and hence algebraic K-theory in good cases, it is sufficient to understand $\mathrm{TR}^n(A;p)$ and $R, F : \mathrm{TR}^{n+1}_*(A;p) \to \mathrm{TR}^n_*(A;p)$ for each p and n . The homotopy groups of these spectra are denoted

$$\mathrm{TR}^n_q(A;p) = [S^q \wedge S^1/C_{p^{n-1}+}, T(A)]_{S^1}.$$

One class of singular rings for which the algebraic K-theory is particularly approachable is pointed monoid algebras, $A(\Pi)$. This approach was first used by Hesselholt and Madsen [12] to compute the algebraic K-theory of $\mathbb{F}_p[x]/x^m$. To compute the K-theory of $A(\Pi)$ using the approach outlined above, one first needs to understand the topological Hochschild homology $T(A(\Pi))$. Hesselholt and Madsen [13] proved that there is an equivalence of S^1-spectra

$$T(A(\Pi)) \simeq T(A) \wedge B^{cy}(\Pi),$$

where $B^{cy}(\Pi)$ denotes the cyclic bar construction on the pointed monoid Π. As above, trace methods essentially reduce the computation of $K_q(A(\Pi))$ to that of

$$\mathrm{TR}^n_q(A(\Pi);p) = \pi_q(T(A(\Pi))^{C_{p^{n-1}}}) = [S^q \wedge S^1/C_{p^{n-1}+}, T(A(\Pi))]_{S^1}.$$

Using the S^1 equivalence of spectra above we can rewrite this as

$$\mathrm{TR}^n_q(A(\Pi);p) = [S^q \wedge S^1/C_{p^{n-1}+}, T(A) \wedge B^{cy}(\Pi)]_{S^1}.$$

If one can understand how $B^{cy}(\Pi)$ is built out of S^1-representation spheres, this gives a formula for these TR-groups in terms of groups of the form

$$\mathrm{TR}^n_{q-\lambda}(A;p) = [S^q \wedge S^1/C_{p^{n-1}+}, T(A) \wedge S^\lambda]_{S^1}.$$

Here λ is a finite-dimensional S^1-representation and S^λ denotes the one-point compactification of this representation. These groups are $RO(S^1)$-graded equivariant homotopy groups of the S^1-spectrum $T(A)$. Recall that $RO(S^1)$ is the ring of virtual real representations of S^1, meaning that an element $\alpha \in RO(S^1)$ can be written as

$$\alpha = [\beta] - [\gamma]$$

where β and γ are finite-dimensional real S^1-representations. For $\alpha = [\beta] - [\gamma]$ in $RO(S^1)$ the TR-group $\mathrm{TR}^n_\alpha(A;p)$ is defined by

$$\mathrm{TR}^n_\alpha(A;p) = \pi_\alpha T(A)^{C_{p^{n-1}}} = [S^\beta \wedge S^1/C_{p^{n-1}+}, S^\gamma \wedge T(A)]_{S^1},$$

generalizing the ordinary TR-groups. As demonstrated above, these $RO(S^1)$-graded TR-groups arise naturally in the computation of the algebraic K-theory of some singular rings. Indeed, in some cases the computation of the algebraic K-theory groups $K_q(A(\Pi))$ can be reduced to the computation of the $RO(S^1)$-graded TR-groups $\mathrm{TR}^n_{q-\lambda}(A;p)$. However, few computations have been done of these $RO(S^1)$-graded TR-groups. The groups $\mathrm{TR}^n_\alpha(A;p)$ were only known in general when $A = \mathbb{F}_p$ and the dimensional of α was even [9]. The current paper broadly extends what is known about $RO(S^1)$-graded TR-groups, making computations for $A = \mathbb{F}_p, \mathbb{Z}$, and ℓ.

This paper serves two main purposes. First, we use the results of this paper in [2], which is joint work with Lars Hesselholt, to compute the relative K-groups $K_*(\mathbb{Z}[x]/(x^m),(x))$ up to extensions, and in [1] to compute the relative K-groups $K_*(\mathbb{Z}[x,y]/(xy),(x,y))$ up to extensions. Theorem 1.1 below is the necessary input to the trace method approach described above, allowing us to make such computations. From this input we compute the relative TC-groups $\mathrm{TC}_*(\mathbb{Z}[x]/(x^m),(x);p,\mathbb{Z}/p)$ and $\mathrm{TC}_*(\mathbb{Z}[x,y]/(xy),(x,y);p,\mathbb{Z}/p)$. Combined with a rational computation this tells us the rank and the number of torsion summands in each degree, and in particular that $\mathrm{TC}_{2i+1}(\mathbb{Z}[x]/(x^m),(x);p) \cong \mathbb{Z}^{m-1}$ and $\mathrm{TC}_{2i}(\mathbb{Z}[x,y]/(xy),(x,y);p) \cong \mathbb{Z}$ are torsion free. An Euler characteristic argument then gives the order of the torsion groups.

The second purpose of the current paper is to try to understand the algebraic structure satisfied by the $RO(S^1)$-graded TR-groups. The algebraic structure satisfied by the ordinary ($\mathbb{Z}$-graded) TR-groups is very rigid, and this has proved very useful [13, 14], e.g. by considering the universal example. A better understanding of the algebraic structure of the $RO(S^1)$-graded TR-groups should be similarly useful. While we do not yet have a complete understanding of this algebraic structure, we hope that having some computations will help in this regard.

Note that in cases where computing $\mathrm{TR}^n_*(A;p)$ with integral coefficients proves to be too difficult one can instead consider the groups $\mathrm{TR}^n_*(A;p,V) = \pi_*(T(A)^{C_{p^{n-1}}} \wedge V)$ for a suitable finite complex V. For instance, smashing with the mod p Moore spectrum $V(0) = S/p$ was used in [6] to compute the mod p groups $\mathrm{TR}^n_*(\mathbb{Z};p,V(0)) = \mathrm{TR}^n_*(\mathbb{Z};p,\mathbb{Z}/p)$, and smashing with the Smith-Toda complex $V(1) = S/(p,v_1)$ was used in [4] to compute $\mathrm{TR}^n_*(\ell;p,V(1))$, the TR-groups of the Adams summand ℓ of connective complex K-theory localized at a prime p, with $V(1)$-coefficients. In both of these cases, the $*$ refers to an integer grading. We will use this technique of smashing with a finite complex in our computations, which are $RO(S^1)$-graded.

In this paper we calculate $\mathrm{TR}^n_\alpha(\mathbb{F}_p;p)$, the $RO(S^1)$-graded TR-groups of $\mathbb{F}_p$, $\mathrm{TR}^n_\alpha(\mathbb{Z};p,V(0))$, the $RO(S^1)$-graded TR-groups of $\mathbb{Z}$ with mod p coefficients, and $\mathrm{TR}^n_\alpha(\ell;p,V(1))$, the $RO(S^1)$-graded TR-groups of ℓ with $V(1)$ coefficients. The calculations in these three cases are essentially identical. To treat all three cases simultaneously, we introduce an integer $c \geq 0$, the chromatic level. If $c = 0$ we let $A = \mathbb{F}_p$ and use integral coefficients. If $c = 1$ we let $A = \mathbb{Z}$ and use mod p coefficients. If $c = 2$ we let $A = \ell$ and use $V(1)$-coefficients. Given a prime p such that the spectrum $BP\langle c\rangle$ with coefficients $\mathbb{Z}_{(p)}[v_1, \ldots, v_c]$ (or its p-completion) is E_∞ and the Smith-Toda complex $V(c)$ exists and is a ring spectrum, the obvious generalization of the calculations in the paper applies.

To state some of these results, we must first introduce some notation. Given a virtual real representation $\alpha \in RO(S^1)$, we define a prime operation by $\alpha' = \rho_p^* \alpha^{C_p}$, where $\rho_p : S^1 \to S^1/C_p$ is the isomorphism given by the p'th root [9]. We let $\alpha^{(n)}$ denote the n-fold iterated prime operation applied to α. A real S^1-representation can be decomposed as a direct sum of copies of the trivial representation $\mathbb{R}$ and the 2-dimensional representations $\mathbb{C}(n)$ with action given by $\lambda \cdot z = \lambda^n z$ for $n \geq 1$. The prime operation acts on these summands as follows:

$$\mathbb{C}(n)' = \begin{cases} \mathbb{C}(\frac{n}{p}) & \text{if } p \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

and $\mathbb{R}' = \mathbb{R}$.

Given a virtual real representation μ, we often write $\mu = \alpha + q$ as a sum of a complex representation $\alpha \in R(S^1)$, and a trivial representation $q \in \mathbb{Z}$. Let $d_i(\alpha) = \dim_{\mathbb{C}}(\alpha^{(i)})$. The $RO(S^1)$-graded TR-groups considered in this paper all have the property that $\mathrm{TR}^n_{\alpha+*}$ for $* \in \mathbb{Z}$ is determined by the sequence of integers

$$d_0(\alpha), \ldots, d_{n-1}(\alpha).$$

Given any sequence of integers $d_0, \ldots, d_{n-1}$ it is possible to find a virtual representation α with $d_i = d_i(\alpha)$ for each i. If $\alpha = \lambda$ or $\alpha = -\lambda$ for an actual representation λ, this sequence of integers is non-increasing or non-decreasing, respectively, and the TR-calculations simplify.

Fix an integer $c \in \{0, 1, 2\}$, and define

$$\delta^n_c(\alpha) = -d_0(\alpha) + \sum_{1 \le k \le n-1} \big[d_{k-1}(\alpha) - d_k(\alpha)\big] p^{ck} \tag{1}$$

If $c = 0$, let $A = \mathbb{F}_p$ and $V = S^0$. If $c = 1$, let $A = \mathbb{Z}$ and $V = V(0)$. If $c = 2$, let $A = \ell$ and $V = V(1)$. We prove in Theorem 4.1 below that in the stable range, i.e., for q sufficiently large, we have

$$\mathrm{TR}^n_{\alpha+q}(A; p, V) \cong \mathrm{TR}^n_{q-2\delta^n_c(\alpha)}(A; p, V).$$

We highlight the following result, which is essential to the K-theory computations in [2] and [1]:

Theorem 1.1. *Let λ be a finite complex S^1-representation. Then for any prime p the finite $\mathbb{Z}_{(p)}$-modules $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z}; p, \mathbb{Z}/p)$ have the following structure:*

1. *For $q \ge 2\dim_{\mathbb{C}}(\lambda)$, $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z}; p, \mathbb{Z}/p)$ has length n, if q is congruent to $2\delta^n_1(\lambda)$ or $2\delta^n_1(\lambda) - 1$ modulo $2p^n$, and $n - 1$ otherwise.*
2. *For $2\dim_{\mathbb{C}}(\lambda^{(s)}) \le q < 2\dim_{\mathbb{C}}(\lambda^{(s-1)})$ with $1 \le s < n$, $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z}; p, \mathbb{Z}/p)$ has length $n - s$ if q is congruent to $2\delta^{n-s}_1(\lambda^{(s)})$ or $2\delta^{n-s}_1(\lambda^{(s)}) - 1$ modulo $2p^{n-s}$ and $n - s - 1$, otherwise.*
3. *For $q < 2\dim_{\mathbb{C}}(\lambda^{(n-1)})$, $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z}; p, \mathbb{Z}/p)$ is zero.*

At an odd prime p, $\mathrm{TR}^n_\alpha(\mathbb{Z}; p, \mathbb{Z}/p)$ is automatically a $\mathbb{Z}/p$-vector space. It follows a posteriori that $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z}; 2, \mathbb{Z}/2)$ is a $\mathbb{Z}/2$-vector space; see [2, Corollary 2.7].

1.1. **Organization.** We begin in §2 by introducing the fundamental diagram of TR-theory, which will be essential to the computations throughout the paper. In §3 we set up a spectral sequence from the homotopy groups of a homotopy orbit spectrum to the TR-groups we are aiming to compute. In §4 we study the Tate spectral sequence in the $RO(S^1)$-graded setting, which is essential to understanding the homotopy orbit spectrum which serves as input for our computations. We handle the cases of $\mathbb{F}_p$, $\mathbb{Z}$, and ℓ simultaneously. We find in Theorem 4.1 below that in each case the Tate spectral sequence is a shifted version of the corresponding $\mathbb{Z}$-graded spectral sequence. In §5 we study the effect of truncating the Tate spectral sequence to the first and second quadrant, obtaining spectral sequences converging to the homotopy orbits and the homotopy fixed points. This provides the induction step needed to prove Theorem 4.1 from the previous section. In §6 we describe the homotopy orbit to TR spectral sequence from §3 in our examples for a general virtual representation α. In §7 we consider the case $A = \mathbb{F}_p$ and use the homotopy orbit to TR spectral sequence with $\mathbb{Z}/p^l$ coefficients for all $l \ge 1$ to give an algorithm for computing $\mathrm{TR}^{n+1}_{\alpha+*}(\mathbb{F}_p; p)$ for any virtual representation α. In §8 we specialize to representations of the form $-\lambda$, where λ is an actual S^1-representation. We show that in this case the homotopy orbit to TR spectral sequence simplifies, and prove Theorem 1.1.

2. The fundamental diagram

The TR-groups are connected by several operators: R, F, V and d. In the ordinary (integer-graded) case, there are maps as follows (see [15] for more details). Inclusion of fixed points induces a map

$$F : \mathrm{TR}^{n+1}_q(A; p) \to \mathrm{TR}^n_q(A; p)$$

called the Frobenius. This map has an associated transfer,

$$V : \mathrm{TR}^n_q(A;p) \to \mathrm{TR}^{n+1}_q(A;p),$$

the Verschiebung. The differential

$$d : \mathrm{TR}^n_q(A;p) \to \mathrm{TR}^n_{q+1}(A;p)$$

is given by multiplying with the fundamental class of S^1 using the circle action. Topological Hochschild homology is a cyclotomic spectrum [13], which gives a map

$$R : \mathrm{TR}^{n+1}_q(A;p) \to \mathrm{TR}^n_q(A;p)$$

called the restriction. The identification of the target of the restriction map with $\mathrm{TR}^n(A;p)$ uses this cyclotomic structure of $T(A)$, which identifies the geometric fixed points $T(A)^{gC_p}$ with $T(A)$. This identification uses a change of universe functor. As a special case, consider $T(G)$ for G a topological group. Then $T(G) \simeq \Sigma^\infty Hom(S^1, BG)$ is the suspension spectrum of the free loop space on BG and the geometric fixed points $T(G)^{gC_p} \simeq \Sigma^\infty Hom(S^1/C_p, BG)$ is the free loop space on loops parametrized by S^1/C_p.

The primary approach used to compute TR-groups is to compare the fixed point spectra to the homotopy fixed point spectra. Let E denote a free contractible S^1-space. Recall that the homotopy fixed point spectrum is defined by $T(A)^{hC_{p^n}} := F(E_+, T(A))^{C_{p^n}}$, and the TR-spectrum is defined by $\mathrm{TR}^{n+1}(A;p) := T(A)^{C_{p^n}}$. The map $E_+ \to S^0$ given by projection onto the non-basepoint induces a map

$$\Gamma_n : \mathrm{TR}^{n+1}(A;p) \to T(A)^{hC_{p^n}}$$

The general strategy for computing the homotopy groups $\mathrm{TR}^{n+1}_q(A;p)$ is to compute $\pi_q(T(A)^{hC_{p^n}})$ and the map Γ_n. This is facilitated through the use of a fundamental diagram of horizontal cofiber sequences [13]:

$$\begin{tikzcd} T(A)_{hC_{p^n}} \arrow[r, "N"] \arrow[d, "="] & \mathrm{TR}^{n+1}(A;p) \arrow[r, "R"] \arrow[d, "\Gamma_n"] & \mathrm{TR}^n(A;p) \arrow[r, "\partial"] \arrow[d, "\hat{\Gamma}_n"] & \Sigma T(A)_{hC_{p^n}} \arrow[d, "="] \\ T(A)_{hC_{p^n}} \arrow[r, "N^h"] & T(A)^{hC_{p^n}} \arrow[r, "R^h"] & T(A)^{tC_{p^n}} \arrow[r, "\partial"] & \Sigma T(A)_{hC_{p^n}} \end{tikzcd} \tag{2}$$

Let $\tilde{E}$ denote the cofiber of $E_+ \to S^0$. Then $T(A)_{hC_{p^n}} := (E_+ \wedge T(A))^{C_{p^n}}$ is the homotopy orbit spectrum and $T(A)^{tC_{p^n}} := (\tilde{E} \wedge F(E_+, T(A))^{C_{p^n}}$ is the Tate spectrum, see [10]. Results of Tsalidis [18] characterize situations when this map Γ_n is an isomorphism.

The computation of $RO(S^1)$-graded TR-groups can be approached similarly. As before, we have the Frobenius $F : \mathrm{TR}^{n+1}_\alpha(A;p) \to \mathrm{TR}^n_\alpha(A;p)$, the Verschiebung $V : \mathrm{TR}^n_\alpha(A;p) \to \mathrm{TR}^{n+1}_\alpha(A;p)$, the differential $d : \mathrm{TR}^n_\alpha(A;p) \to \mathrm{TR}^n_{\alpha+1}(A;p)$, and the restriction $R : \mathrm{TR}^{n+1}_\alpha(A;p) \to \mathrm{TR}^n_{\alpha'}(A;p)$. Note that the target of R is the group in dimension α', not α (see [13] for a detailed explanation of the restriction in this context).

The fundamental diagram also extends to this $RO(S^1)$-graded context. Let T denote $T(A)$ and let $T[\alpha] = T(A) \wedge S^{-\alpha}$ denote the desuspension of T by α. Then we have the following fundamental diagram of horizontal cofiber sequences [13]:

$$
\begin{tikzcd}
T[\alpha]_{hC_{p^n}} \arrow[r, "N"] \arrow[d, "="] & \mathrm{TR}^{n+1}(A;p)[\alpha] \arrow[r, "R"] \arrow[d, "\Gamma_n"] & \mathrm{TR}^n(A;p)[\alpha'] \arrow[r, "\partial"] \arrow[d, "\hat{\Gamma}_n"] & \Sigma T[\alpha]_{hC_{p^n}} \arrow[d, "="] \\
T[\alpha]_{hC_{p^n}} \arrow[r, "N^h"] & T[\alpha]^{hC_{p^n}} \arrow[r, "R^h"] & T[\alpha]^{tC_{p^n}} \arrow[r, "\partial"] & \Sigma T[\alpha]_{hC_{p^n}}
\end{tikzcd} \tag{3}
$$

Notice that $\mathrm{TR}^n(A;p)[\alpha']$ appears, rather than $\mathrm{TR}^n(A;p)[\alpha]$. We can take homotopy groups of the top row and get a long exact sequence

$$\ldots \to \pi_q T[\alpha]_{hC_{p^n}} \to \mathrm{TR}^{n+1}_{\alpha+q}(A;p) \to \mathrm{TR}^n_{\alpha'+q}(A;p) \to \pi_{q-1} T[\alpha]_{hC_{p^n}} \to \ldots \tag{4}$$

This is the fundamental long exact sequence of $RO(S^1)$-graded TR-theory. The strategy for computing $\mathrm{TR}^n_{\alpha+*}(A;p)$ is to use Diagram 3 and induction. One can attempt to understand the bottom row via spectral sequences, see [10]. For instance, the Tate spectral sequence computes $\pi_q(T[\alpha]^{tC_{p^n}})$:

$$E^2_{s,t} = \hat{H}^{-s}(C_{p^{n-1}}, \pi_t(T[\alpha])) \Rightarrow \pi_{s+t}(T[\alpha]^{tC_{p^n}})$$

Restrictions of this spectral sequence give spectral sequences converging to the homotopy groups of the homotopy fixed points and homotopy orbits. We use these spectral sequences to make computations of the homotopy groups on the bottom of the diagram. Understanding the maps Γ_n and $\hat{\Gamma}_n$ is also key to our arguments. Theorem 5.1 below, which is due to Tsalidis [18] in the non-equivariant case, says that if $\hat{\Gamma}_1$ is an isomorphism in sufficiently high degrees then so are Γ_n and $\hat{\Gamma}_n$ for all n. If we know $\mathrm{TR}^n_{\alpha'+*}$ we can then use $\hat{\Gamma}_n$ to understand the Tate spectrum $T[\alpha]^{tC_{p^n}}$ and the rest of the bottom row. This gives $\mathrm{TR}^{n+1}_{\alpha+*}$ in sufficiently high degrees. We are then left to compute $\mathrm{TR}^{n+1}_{\alpha+*}$ in the unstable range. In the following section we develop a spectral sequence that allows us to do the computations in the unstable range. This spectral sequence starts from the homotopy orbits and converges to the TR-groups we would like to compute. The spectral sequence allows us to treat the cases of $\mathbb{F}_p$, $\mathbb{Z}$, and ℓ simultaneously. However, in the case of $\mathbb{F}_p$ there are additional extension issues which need to be resolved.

In the $\mathbb{Z}$-graded case, it is useful to first compute $\mathrm{TR}^n_*(\mathbb{F}_p;p,\mathbb{Z}/p)$. This shows that $\mathrm{TR}^n_{2q}(\mathbb{F}_p;p)$ is cyclic and $\mathrm{TR}^n_{2q+1}(\mathbb{F}_p;p) = 0$, and from this we conclude that the relevant extensions are maximally nontrivial. In the $RO(S^1)$-graded case, $\mathrm{TR}^n_{\alpha+q}(\mathbb{F}_p;p)$ could have several summands, and indeed, for many α it does. It is possible to compute the order of $\mathrm{TR}^n_{\alpha+q}(\mathbb{F}_p;p)$ inductively using Diagram 3, and computations with $\mathbb{Z}/p$-coefficients determine the number of summands, but this information is not enough to determine the group. We solve this problem by using $\mathbb{Z}/p^l$ coefficients for all $l \geq 1$, calculating the associated graded of $\mathrm{TR}^n_{\alpha+q}(\mathbb{F}_p;p,\mathbb{Z}/p^l)$, and this is enough to solve the extension problem.

No such extension problems arise in our computations of $\mathrm{TR}^n_{\alpha+*}(\mathbb{Z};p,V(0))$ and $\mathrm{TR}^n_{\alpha+*}(\ell;p,V(1))$ as graded abelian groups. However, it can be convenient to consider these not only as abelian groups but as a module over $\mathbb{F}_p[v_1]$ using the map $v_1 : \Sigma^{2p-2}V(0) \to V(0)$ in the first case and a module over $\mathbb{F}_p[v_2]$ using the map $v_2 : \Sigma^{2p^2-2}V(1) \to V(1)$ in the second case. This simplifies the bookkeeping, and by writing $\mathbb{Z}/p^n$ as $\mathbb{F}_p[v_0]/v_0^n$ we can treat all three cases simultaneously. In the stable range the module structure over $\mathbb{F}_p[v_c]$ is clear, but there could be hidden v_c-multiplications in low degree. One could then consider using $S(p,v_1^l)$ or

$S/(p, v_1, v_2^l)$ as coefficients, and although we believe this would give a similar algorithm for resolving the extensions as the one we find for $\mathrm{TR}^n_{\alpha+*}(\mathbb{F}_p; p)$ we will not pursue that avenue here. We will express $\mathrm{TR}^n_{\alpha+*}(\mathbb{Z}; p, V(0))$ as an $\mathbb{F}_p[v_1]$-module and $\mathrm{TR}^n_{\alpha+*}(\ell; p, V(1))$ as an $\mathbb{F}_p[v_2]$-module, with the caveat that there might be additional hidden extensions.

Before we proceed a comment on the dependency of the prime p is in order. The mod 2 Moore spectrum $V(0)$ is not a ring spectrum [3], so $\mathrm{TR}^n_*(\mathbb{Z}; 2, V(0))$ has no ring structure. This can be worked around by considering $V(0)$ as a module over the mod 4 Moore spectrum. Indeed, Rognes [17] has shown that the Tate spectral sequence converging to $V(0)_* T(\mathbb{Z})^{tC_{2^n}}$, and hence also the corresponding homotopy fixed point and homotopy orbit spectral sequences, behave the same way as for odd primes. In particular the differentials are derivations with respect to a formal algebra structure on the spectral sequence and the pattern of differentials is the same as for odd primes. This immediately carries over to the $RO(S^1)$-graded case.

There is also no way to define multiplication by v_1 on $V(0)$ at $p = 2$, so we cannot ask about the module structure over $\mathbb{F}_2[v_1]$. Hence the results about $\mathrm{TR}^n_{\alpha+*}(\mathbb{Z}; 2, V(0))$ have to be interpreted additively only, or perhaps as a module over $\mathrm{TR}^n_*(\mathbb{Z}; 2, \mathbb{Z}/4)$. The Smith-Toda complex $V(1)$ does not exist at $p = 2$, and although one could use coefficients such as $S/(2, v_1^4)$ or $S/(4, v_1^4)$ to study the TR groups of ku and ko we will not pursue that here.

At $p = 3$, $V(1)$ is not a ring spectrum, so our results for $\mathrm{TR}^n_{\alpha+*}(\ell; p, V(1))$ are only valid for primes $p \geq 5$. We conjecture that the situation for $\mathrm{TR}^n_{\alpha+q}(\ell; p, V(1))$ at $p = 3$ is analogous to the one for $\mathrm{TR}^n_{\alpha+*}(\mathbb{Z}; p, V(0))$ at $p = 2$ where $S/(3, v_1^3)$ plays the role of the mod 4 Moore spectrum, though we will not study that question here.

3. The homotopy orbit to TR spectral sequence

It is possible to glue together the long exact sequences in Equation 4 to obtain a spectral sequence converging to $\mathrm{TR}^{n+1}_{\alpha+*}(A; p, V)$ with coefficients in V. For this section A can be any connective S-algebra and V can be any spectrum. Let $T = T(A)$. The E_1 term is given by

$$E_1^{s,t}(\alpha) = \begin{cases} V_t T[\alpha^{(n-s)}]_{hC_{p^s}} & \text{for } 0 \leq s \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

This spectral sequence converges to $\mathrm{TR}^{n+1}_{\alpha+t}(A; p, V)$.

The reason this spectral sequence has not been introduced before is that in previously computed examples, one can understand $\mathrm{TR}^{n+1}_*(A; p, V)$ completely by comparing with $V_* T^{hC_{p^n}}$. In the $RO(S^1)$-graded case, there is a range of degrees where this comparison is less useful.

The d_r differential has bidegree $(r, -1)$,

$$d_r : E_r^{s,t}(\alpha) \to E_r^{s+r,t-1}(\alpha),$$

and can be defined as follows: For $x \in V_t T[\alpha^{(n-s)}]_{hC_{p^s}}$, $d_r(x)$ is given by lifting $N(x)$ up to $\mathrm{TR}^{s+r}_{\alpha^{(n-s-r+1)}+t}(A;p,V)$ and then applying ∂:

$$\begin{tikzcd}
 & \mathrm{TR}^{s+r}_{\alpha^{(n-s-r+1)}+t}(A;p,V) \arrow[r, "\partial"] \arrow[d, "R"] & V_{t-1}T[\alpha^{(n-s-r)}]_{hC_{p^{s+r}}} \\
 & \vdots \arrow[d, "R"] & \\
 & \mathrm{TR}^{s+2}_{\alpha^{(n-s-1)}+t}(A;p,V) \arrow[r, "\partial"] \arrow[d, "R"] & V_{t-1}T[\alpha^{(n-s-2)}]_{hC_{p^{s+2}}} \\
V_t T[\alpha^{(n-s)}]_{hC_{p^s}} \arrow[r, "N"] & \mathrm{TR}^{s+1}_{\alpha^{(n-s)}+t}(A;p,V) \arrow[r, "\partial"] & V_{t-1}T[\alpha^{(n-s-1)}]_{hC_{p^{s+1}}}
\end{tikzcd}$$

Observation 3.1. *We note that if A and V are (-1)-connected the filtration s piece $V_*T[\alpha^{(n-s)}]_{hC_{p^s}}$ is zero in degree $* < -2d_{n-s}(\alpha)$.*

Definition 3.2. *Consider the short exact sequence*

$$0 \to coker(R^h)[-1] \to V_*T[\alpha]_{hC_{p^n}} \to ker(R^h) \to 0$$

obtained by taking $V_*(-)$ of the bottom row of Diagram 3. We call the image of $coker(R^h)[-1]$ in $V_*T[\alpha]_{hC_{p^n}}$ the Tate piece and denote it by $V_*^tT[\alpha]_{hC_{p^n}}$. If the sequence is split we choose a splitting and call the image of $ker(R^h)$ under the splitting the homotopy fixed point piece, denoted $V_*^hT[\alpha]_{hC_{p^n}}$.

Hence if the above short exact sequence splits we have a decomposition

$$V_*T[\alpha]_{hC_{p^n}} \cong V_*^tT[\alpha]_{hC_{p^n}} \oplus V_*^hT[\alpha]_{hC_{p^n}}.$$

The purpose of the above definition is to get a better handle on the differentials in the homotopy orbit to TR spectral sequence:

Lemma 3.3. *In the homotopy orbit to TR spectral sequence, every class in the Tate piece $V_*^tT[\alpha^{(n-s)}]_{hC_{p^s}}$ is a permanent cycle, and the image of any differential is contained in the Tate piece. If the short exact sequence in Definition 3.2 splits then all differentials go from a subgroup of the homotopy fixed point piece to a quotient of the Tate piece.*

Proof. This is a straightforward diagram chase, using the construction of the spectral sequence and Diagram 3. □

We will denote classes in $V_*^tT[\alpha]_{hC_{p^n}}$ by their name in $V_*T[\alpha]^{tC_{p^n}}$ and classes in $V_*^hT[\alpha]_{hC_{p^n}}$ by their name in $V_*T[\alpha]^{hC_{p^n}}$.

4. The Tate spectral sequence

In order to use the spectral sequence from the previous section, we must first understand the homotopy orbit spectrum. The homotopy orbit spectral sequence computing $V_*T[\alpha]_{hC_{p^n}}$ is the restriction of the corresponding Tate spectral sequence to the first quadrant, so we first study the Tate spectral sequence converging to $V_*T[\alpha]^{tC_{p^n}}$. For $c \in \{0,1,2\}$, let V and A be as in the introduction, and let $T = T(A)$.

Recall [13, 6, 4] that the homotopy groups of topological Hochschild homology with these coefficients are given as follows:

$$\begin{aligned} \pi_*T(\mathbb{F}_p) &= P(\mu_0), \\ V(0)_*T(\mathbb{Z}) &= E(\lambda_1)\otimes P(\mu_1), \\ V(1)_*T(\ell) &= E(\lambda_1,\lambda_2)\otimes P(\mu_2). \end{aligned}$$

Here $P(-)$ denotes a polynomial algebra and $E(-)$ denotes an exterior algebra, both over $\mathbb{F}_p$. The degrees are given by $|\lambda_i| = 2p^i - 1$ and $|\mu_c| = 2p^c$, with λ_i represented by $\sigma\bar{\xi}_i$ and μ_c represented by $\sigma\bar{\tau}_c$ in the Bökstedt spectral sequence. At $p = 2$, λ_i is represented by $\sigma\bar{\xi}_i^2$ and μ_c is represented by $\sigma\bar{\xi}_{c+1}$.

The above formula for $V(0)_*T(\mathbb{Z})$ can be interpreted multiplicatively even though $V(0)$ is not a ring spectrum at $p = 2$, by using that $V(0)\wedge T(\mathbb{Z}) \simeq T(\mathbb{Z};\mathbb{Z}/2)$. At $p = 3$ we can use that $V(1)\wedge T(\ell) \simeq T(\ell;\mathbb{Z}/3)$ to put a ring structure on $V(1)_*T(\ell)$. (This gives an interpretation of $V(1)_*T(\ell)$ even at $p = 2$.) But note that there is no S^1-action on topological Hochschild homology with coefficients in a bimodule, so there is no corresponding ring structure on the TR-groups if the coefficient spectrum is not a ring spectrum. Rognes [17] has shown that the Tate spectral sequence converging to $V(0)_*T(\mathbb{Z})^{tC_{p^n}}$ has a formal algebra structure, so we can proceed as if $V(0)$ was a ring spectrum.

We have

$$V_*T[\alpha] \cong V_{2d_0(\alpha)+*}T,$$

and we will write this as

$$V_*T[\alpha] = t^{d_0(\alpha)}V_*T \tag{5}$$

where $|t| = -2$.

The Tate spectral sequence converging to $V_*T[\alpha]^{tC_{p^n}}$ has E_2 term given by the Tate cohomology

$$\hat{E}_2(\alpha) = \hat{H}^*(C_{p^n}; V_*T[\alpha]) \cong V_*T(A)\otimes P(t,t^{-1})\otimes E(u_n)[\alpha],$$

a free module over the corresponding non-equivariant spectral sequence on a generator $[\alpha]$. Here $|u_n| = -1$ and $|t| = -2$. The class $v_c \in \pi_{2p^c-2}V$ (recall that $v_0 = p$) maps to a class in $V_*T^{hS^1}$ represented by $t\mu_c$ in the E_2 term of the homotopy fixed point spectral sequence (see e.g. [4, Proposition 4.8]), so by abuse of notation we will denote the class $t\mu_c$ in the C_{p^n} Tate spectral sequence by v_c.

Recall [13, 6, 4] that $V_*T^{tC_{p^n}}$ is $2p^n$-periodic and the definition of $\delta_c^n(\alpha)$ in Equation 1 in the introduction.

Theorem 4.1. *The $RO(S^1)$-graded* TR *groups of A satisfy*

$$\mathrm{TR}^n_{\alpha+*}(A;p,V) \cong \mathrm{TR}^n_{*-2\delta_c^n(\alpha)}(A;p,V)$$

for $*$ sufficiently large, and the V-homotopy groups of $T[\alpha]^{tC_{p^n}}$ satisfy

$$V_*T[\alpha]^{tC_{p^n}} \cong V_{*-2\delta_c^n(\alpha')}T^{tC_{p^n}}$$

for all $*$.

We prove this theorem in the next section, after analyzing the restriction of the Tate spectral sequence to the first and second quadrant.

We spell out the behavior of the Tate spectral sequence in each case. Define $r(n)$ by

$$r(n) = \sum_{1 \le k \le n} p^{ck}. \tag{6}$$

As in the non-equivariant case the classes λ_i and v_c are permanent cycles, and the Tate spectral sequence is determined by the following (compare [13, 6, 4]):

In each case we have a family of differentials given by

$$d_{2r(n)+1}(t^{-k}u_n[\alpha]) = v_c^{r(n-1)+1} t^{p^{cn}-k}[\alpha]$$

if $\nu_p(k - \delta_c^n(\alpha')) \ge cn$. If $c = 0$ this condition is empty, and this is the only family of differentials.

For $c \ge 1$ we have, for each $1 \le j \le n$, a differential

$$d_{2r(j)}(t^{-k}[\alpha]) = v_c^{r(j-1)} t^{p^{cj}-k} \lambda_c[\alpha]$$

if $\nu_p(k - \delta_c^n(\alpha')) = cj - 1$.

Finally, if $c = 2$ we have, for each $1 \le j \le n$, a differential

$$d_{2r(j)/p}(t^{-k}[\alpha]) = v_2^{r(j-1)/p} t^{p^{2j-1}-k} \lambda_1[\alpha]$$

if $\nu_p(k - \delta_2^n(\alpha')) = 2j - 2$.

5. The homotopy orbit and homotopy fixed point spectra

To find $V_*T[\alpha]_{hC_{p^n}}$ and $V_*T[\alpha]^{hC_{p^n}}$ we restrict the Tate spectral sequence from the previous section to the first or second quadrant. Because of our grading conventions, in particular Equation 5 above, the first quadrant means filtration greater than $-2d_0(\alpha)$. Hence the homotopy orbit spectral sequence has E_2-term

$$V_*T \otimes E(u_n)\{t^k[\alpha] \,:\, k < d_0(\alpha)\}[-1]$$

and the homotopy fixed point spectral sequence has E_2-term

$$V_*T \otimes E(u_n)\{t^k[\alpha] \,:\, k \ge d_0(\alpha)\}.$$

Analyzing these spectral sequences is straightforward, but requires some amount of bookkeeping. We will write down $V_*T[\alpha]_{hC_{p^n}}$ completely because it is the input to the homotopy orbit to TR spectral sequence. We will partially describe $V_*T[\alpha]^{hC_{p^n}}$ by explaining how some v_c-towers in the homotopy fixed point piece of $V_*T[\alpha]_{hC_{p^n}}$ become divisible by some power of v_c in $V_*T[\alpha]^{hC_{p^n}}$. The rest of $V_*T[\alpha]^{hC_{p^n}}$ consists of those v_c-towers that are concentrated in negative total degree, and these are isomorphic to the corresponding v_c-towers in $V_*T[\alpha]^{tC_{p^n}}$.

We separate $V_*T[\alpha]_{hC_{p^n}}$ into the Tate piece and the homotopy fixed point piece as in Definition 3.2, and each piece comes in $c+1$ families, each of which can be split into a stable part and an unstable part. In sufficiently high degrees the map R^h in Diagram 3 is zero, so N^h is an isomorphism between the homotopy fixed point piece of $V_*T[\alpha]_{hC_{p^n}}$ and $V_*T[\alpha]^{hC_{p^n}}$ in the stable range. This isomorphism can be described in terms of those differentials in the Tate spectral sequence which go from the first to the second quadrant. Such a differential leaves one class in $V_*T[\alpha]_{hC_{p^n}}$ and one class in $V_*T[\alpha]^{hC_{p^n}}$, neither of which has a corresponding class in $V_*T[\alpha]^{tC_{p^n}}$.

To describe the first family, which is the one "created" by the longest differential $d_{2r(n)+1}$ in the Tate spectral sequence, let $E = \mathbb{F}_p$ for $c = 0$, $E(\lambda_1)$ for $c = 1$ and $E(\lambda_1, \lambda_2)$ for $c = 2$. Then the Tate piece of the first family is as follows:

$$\bigoplus_{\substack{k \geq r(n-1)+1-d_0(\alpha) \\ \nu_p(k-\delta_c^n(\alpha')) \geq cn}} E \otimes P_{r(n-1)+1}(v_c)\{t^{-k}[\alpha]\}[-1]$$
$$\bigoplus_{\substack{1 \leq k+d_0(\alpha) \leq r(n-1) \\ \nu_p(k-\delta_c^n(\alpha')) \geq cn}} E \otimes P_{k+d_0(\alpha)}(v_c)\{t^{-k}[\alpha]\}[-1]$$

In particular, in the stable range we have v_c-towers of height $r(n-1)+1$ starting in degree

$$2\delta_c^n(\alpha') + mp^{cn}.$$

Similarly, the homotopy fixed point piece is as follows:

$$\bigoplus_{\substack{k \geq r(n-1)+1 \\ \nu_p(k-d_0(\alpha)-\delta_c^n(\alpha')) \geq cn}} E \otimes P_{r(n)+1}(v_c)\{t^{d_0(\alpha)}\mu_c^k[\alpha]\}$$
$$\bigoplus_{\substack{1 \leq k \leq r(n) \\ \nu_p(k-d_0(\alpha)-\delta_c^n(\alpha')) \geq cn}} E \otimes P_k(v_c)\{v_c^{r(n)+1-k}t^{d_0(\alpha)}\mu_c^{k-p^{cn}}[\alpha]\}$$

In particular, in the stable range we have v_c-towers of height $r(n)+1$ starting in degree

$$-2d_0(\alpha) + 2p^c(d_0(\alpha) + \delta_c^n(\alpha') + mp^{cn}) = 2\delta_c^{n+1}(\alpha) + mp^{c(n+1)}.$$

Next we compare this to $V_*T[\alpha]^{hC_{p^n}}$. For the v_c-towers of maximal height, the map N^h in Diagram 3 is an isomorphism. Now consider a generator x of

$$P_k(v_c)\{v_c^{r(n)+1-k}t^{d_0(\alpha)}\mu_c^{k-p^{cn}}[\alpha]\} = P_k(v_c)\{t^{r(n)+1-k+d_0(\alpha)}\mu_c^{r(n-1)+1}[\alpha]\}$$

and its image $N^h(x)$ in $V_*T[\alpha]^{hC_{p^n}}$. We have two cases, with the first case only applicable if $c \geq 1$. First, if $k < p^{cn}$ then $N^h(x)$ is divisible by $v_c^{r(n-1)+1}$ and we get a v_c-tower

$$E \otimes P_{r(n-1)+1+k}(v_c)\{t^{p^{cn}-k+d_0(\alpha)}[\alpha]\}.$$

If $k \geq p^{cn}$ then $N^h(x)$ is divisible by $v_c^{r(n)+1-k}$ and we get a v_c-tower

$$E \otimes P_{r(n)+1}(v_c)\{t^{d_0(\alpha)}\mu_c^{k-p^{cn}}[\alpha]\}.$$

If $c \geq 1$ the second family is "created" by the differentials $d_{2r(j)}$ for $1 \leq j \leq n$. Let $E'_n = E(u_n)$ if $c = 1$ and $E(\lambda_1, u_n)$ if $c = 2$. Then the Tate piece of the second family is as follows:

$$\bigoplus_{2 \leq j \leq n} \bigoplus_{\substack{k \geq r(j-1)-d_0(\alpha) \\ \nu_p(k-\delta_c^n(\alpha')) = cj-1}} E'_n \otimes P_{r(j-1)}(v_c)\{t^{-k}\lambda_c[\alpha]\}[-1]$$
$$\bigoplus_{2 \leq j \leq n} \bigoplus_{\substack{1 \leq k+d_0(\alpha) \leq r(j-1)-1 \\ \nu_p(k-\delta_c^n(\alpha')) = cj-1}} E'_n \otimes P_{k+d_0(\alpha)}(v_c)\{t^{-k}\lambda_c[\alpha]\}[-1]$$

Similarly, the homotopy fixed point piece is as follows:

$$\bigoplus_{1\le j\le n} \bigoplus_{\substack{k\ge r(j-1)\\ \nu_p(k-d_0(\alpha)-\delta_c^n(\alpha'))=cj-1}} E_n'\otimes P_{r(j)}(v_c)\{t^{d_0(\alpha)}\mu_c^k\lambda_c[\alpha]\}$$
$$\bigoplus_{1\le j\le n} \bigoplus_{\substack{1\le k\le r(j)-1\\ \nu_p(k-d_0(\alpha)-\delta_c^n(\alpha'))=cj-1}} E_n'\otimes P_k(v_c)\{v_c^{r(j)-k}t^{d_0(\alpha)}\mu_c^{k-p^{cj}}\lambda_c[\alpha]\}$$

Consider a generator x of $P_k(v_c)\{v_c^{r(j)-k}t^{d_0(\alpha)}\mu_c^{k-p^{cj}}\lambda_c[\alpha]\}$ and its image $N^h(x)$ in $V_*T[\alpha]^{hC_{p^n}}$. Again we have two cases. If $k<p^{cj}$ then $N^h(x)$ is divisible by $v_c^{r(j-1)}$ and we get a v_c-tower

$$E_n'\otimes P_{r(j-1)+k}(v_c)\{t^{p^{cj}-k+d_0(\alpha)}\lambda_c[\alpha]\}.$$

If $k\ge p^{cj}$ then $N^h(x)$is divisible by $v_c^{r(j)-k}$ and we get a v_c-tower

$$E_n'\otimes P_{r(j)}(v_c)\{t^{d_0(\alpha)}\mu_c^{k-p^{cj}}\lambda_c[\alpha]\}.$$

Finally, if $c=2$ the third family is “created” by the differentials $d_{2r(j)/p}$ for $1\le j\le n$. Let $E_n''=E(\lambda_2,u_n)$. Then the Tate piece of the third family is as follows:

$$\bigoplus_{2\le j\le n} \bigoplus_{\substack{k\ge r(j-1)/p-d_0(\alpha)\\ \nu_p(k-\delta_2^n(\alpha'))=2j-2}} E_n''\otimes P_{r(j-1)/p}(v_2)\{t^{-k}\lambda_1[\alpha]\}[-1]$$
$$\bigoplus_{2\le j\le n} \bigoplus_{\substack{1\le k+d_0(\alpha)\le r(j-1)/p-1\\ \nu_p(k-\delta_2^n(\alpha'))=2j-2}} E_n''\otimes P_{k+d_0(\alpha)}(v_2)\{t^{-k}\lambda_1[\alpha]\}[-1]$$

Similarly, the homotopy fixed point piece is as follows:

$$\bigoplus_{1\le j\le n} \bigoplus_{\substack{k\ge r(j-1)/p\\ \nu_p(k-d_0(\alpha)-\delta_2^n(\alpha'))=2j-2}} E_n''\otimes P_{r(j)/p}(v_2)\{t^{d_0(\alpha)}\mu_2^k\lambda_1[\alpha]\}$$
$$\bigoplus_{1\le j\le n} \bigoplus_{\substack{1\le k\le r(j)/p-1\\ \nu_p(k-d_0(\alpha)-\delta_2^n(\alpha'))=2j-2}} E_n''\otimes P_k(v_2)\{v_2^{r(j)/p-k}t^{d_0(\alpha)}\mu_2^{k-p^{2j-1}}\lambda_1[\alpha]\}$$

Once again, we consider the image $N^h(x)$ of a generator x of the v_2-tower $P_k(v_2)\{v_2^{r(j)/p-k}t^{d_0(\alpha)}\mu_2^{k-p^{2j-1}}\lambda_1[\alpha]\}$ in $V_*T[\alpha]^{hC_{p^n}}$. If $k<p^{2j-1}$ then $N^h(x)$ is divisible by $v_2^{r(j-1)/p}$ and we get a v_2-tower

$$E_n''\otimes P_{r(j-1)/p+k}(v_2)\{t^{p^{2j-1}-k+d_0(\alpha)}\lambda_1[\alpha]\}.$$

If $k\ge p^{2j-1}$ then $N^h(x)$ is divisible by $v_2^{r(j)/p-k}$ and we get a v_2-tower

$$P_{r(j)/p}(v_2)\{t^{d_0(\alpha)}\mu_2^{k-p^{2j-1}}\lambda_1[\alpha]\}.$$

We will use the following theorem, which with integral coefficients is due to Tsalidis [18] in the $\mathbb{Z}$-graded case and Hesselholt-Madsen [13] in the $RO(S^1)$-graded case:

Theorem 5.1. *Suppose the map $\hat{\Gamma}_1 : T(A) \to T(A)^{tC_p}$ induces an isomorphism $V_qT(A) \to V_qT(A)^{tC_p}$ for $q \geq i$. Then, for any $n \geq 1$, $\hat{\Gamma}_n$ induces an isomorphism $\mathrm{TR}^n_{\alpha'+q}(A;p,V) \to V_qT[\alpha]^{tC_{p^n}}$ for*

$$q \geq 2\max(-d_1(\alpha), \dots, -d_n(\alpha)) + i.$$

Equivalently, Γ_n induces an isomorphism $\mathrm{TR}^{n+1}_{\alpha+q}(A;p,V) \to V_qT[\alpha]^{hC_{p^n}}$ in the same range.

The proof with coefficients in V is identical to the proof with integral coefficients. Now we can prove Theorem 4.1:

Proof of Theorem 4.1. In each case Theorem 5.1 applies, see e.g. [4]. For $c = 0$ we have $i = 0$, for $c = 1$ we have $i = 0$ and for $c = 2$ we have $i = 2p-1$ (the class $t^{p^2}\lambda_1\lambda_2$ in $V(1)_*T(\ell)^{tC_p}$ is in degree $2p-2$). Suppose by induction that the statement of the Theorem holds for $\mathrm{TR}^n_{\alpha'+*}(A;p,V)$. Then the map $\hat{\Gamma}_n : \mathrm{TR}^n_{\alpha'+*}(A;p,V) \to V_*T[\alpha]^{tC_{p^n}}$ is coconnective, so $V_*T[\alpha]^{tC_{p^n}}$ is shifted by $2\delta^n_c(\alpha')$ degrees in the stable range. Using that $V_*T[\alpha]^{tC_{p^n}}$ is a module over $V_*T^{tC_{p^n}}$ and that $V_*T^{tC_{p^n}}$ is $2p^n$-periodic the statement for $V_*T[\alpha]^{tC_{p^n}}$ follows. The pattern of differentials in the Tate spectral sequence described after the statement of Theorem 4.1 also follows from this.

Restricting the Tate spectral sequence to the second quadrant gives a spectral sequence computing $V_*T[\alpha]^{hC_{p^n}}$, and each differential on a class t^{-k} in the Tate spectral sequence gives a class

$$t^{d_0(\alpha)}\mu_c^{k+d_0(\alpha)}.$$

The differentials on t^{-k} for various k are shifted by $2\delta^n_c(\alpha')$ degrees, which means that the classes in the homotopy fixed point spectrum are shifted by

$$-2d_0(\alpha) + 2p^c(d_0(\alpha) + \delta^n_c(\alpha')) = 2\delta^{n+1}_c(\alpha)$$

degrees. Using that $\Gamma_n : \mathrm{TR}^{n+1}_{\alpha+*}(A;p,V) \to V_*T[\alpha]^{hC_{p^n}}$ is coconnective the statement then holds for $\mathrm{TR}^{n+1}_{\alpha+*}(A;p,V)$. □

6. A splitting of the homotopy orbit to TR spectral sequence

In this section we describe the homotopy orbit to TR spectral sequence in the three cases of interest. We show that the spectral sequence splits as the direct sum of "small" spectral sequences, with no differentials between different summands.

We first describe the small spectral sequences. Let $E = \mathbb{F}_p$ if $c = 0$, $E(\lambda_1)$ if $c = 1$ and $E(\lambda_1, \lambda_2)$ if $c = 2$. Consider the following diagram:

$$\begin{tikzcd}
 & P_{r(0)+1}(v_c)\{t^{d_n(\alpha)}\mu_c^k\} \arrow[dl, dotted] \arrow[ddl, dotted] \arrow[ddddl, dotted] \arrow[d, no head] \\
P_{r(0)+1}(v_c)\{t^{-p^c k-\delta_c^1(\alpha^{(n)})}\}[-1] \arrow[d, no head] & P_{r(1)+1}(v_c)\{t^{d_{n-1}(\alpha)}\mu_c^{p^c k+d_{n-1}(\alpha)+\delta_c^1(\alpha^{(n)})}\} \arrow[dl, dotted] \arrow[dddl, dotted] \arrow[d, no head] \\
P_{r(1)+1}(v_c)\{t^{-p^{2c} k-\delta_c^2(\alpha^{(n-1)})}\}[-1] \arrow[d, no head] & P_{r(2)+1}(v_c)\{t^{d_{n-2}(\alpha)}\mu_c^{p^{2c} k+d_{n-2}(\alpha)+\delta_c^2(\alpha^{(n-1)})}\} \arrow[ddl, dotted] \arrow[d, no head] \\
\vdots \arrow[d, no head] & \vdots \arrow[d, no head] \\
P_{r(n-1)+1}(v_c)\{t^{-p^{cn} k-\delta_c^n(\alpha')}\}[-1] & P_{r(n)+1}(v_c)\{t^{d_0(\alpha)}\mu_c^{p^{cn} k+d_0(\alpha)+\delta_c^n(\alpha')}\}
\end{tikzcd}$$

For each k, there is a summand of the E_1 term of the homotopy orbit to TR spectral sequence which looks like the above diagram tensored with E (recall that $E = \mathbb{F}_p$, $E(\lambda_1)$ or $E(\lambda_1, \lambda_2)$), with submodules of the modules in the right hand column and quotient modules of the modules in the left hand column (the summands are allowed to be 0). If $c = 0$, this describes the whole E_1 term. If $c = 1$ there is one more family of diagrams to consider and if $c = 2$ there are two more families of diagrams to consider.

For $c = 1$ or 2 the second family of small spectral sequences looks as follows. Recall that $E'_j = E(u_j)$ if $c = 1$ and $E(u_j, \lambda_1)$ if $c = 2$. For each $0 \le j \le n-1$ and each k with $\nu_p(k - d_j(\alpha) + d_{j+1}(\alpha)) = c-1$ we have a corresponding diagram, where the right hand side consists of submodules of

$$E'_{n-j+m} \otimes P_{r(m+1)}(v_c)\{t^{d_{j-m}(\alpha)}\mu_c^{p^{cm}k+d_{j-m}(\alpha)+\delta_c^m(\alpha^{(j-m+1)})}\lambda_c\}$$

for $0 \le m \le j$ and the left hand side consists of quotient modules of

$$E'_{n-j+m} \otimes P_{r(m)}(v_c)\{t^{-p^{cm}k-\delta_c^m(\alpha^{(j-m+1)})}\lambda_c\}[-1]$$

for $1 \le m \le j$.

Finally, if $c = 2$ the third family of small spectral sequences looks as follows. Recall that $E''_j = E(u_j, \lambda_2)$. For each $1 \le j \le n$ and each k with $\nu_p(k - d_j(\alpha) + d_{j+1}(\alpha)) = 0$ we have a corresponding diagram, where the right hand side consists of submodules of

$$E''_{n-j+m} \otimes P_{r(m+1)/p}(v_2)\{t^{d_{j-m}(\alpha)}\mu_2^{p^{2m}k+d_{j-m}(\alpha)+\delta_2^m(\alpha^{(j-m+1)})}\lambda_1\}$$

for $0 \le m \le j$ and the left hand side consists of quotient modules of

$$E''_{n-j+m} \otimes P_{r(m)/p}(v_2)\{t^{-p^{2m}k-\delta_2^m(\alpha^{(j-m+1)})}\lambda_1\}[-1]$$

for $1 \le m \le j$.

The following theorem gives an algorithm for computing the homotopy orbit to TR spectral sequence. The expression for $d_\rho(x)$ looks unpleasant, but for $c = 1$ or 2 the formula, in the case when $d_\rho(x)$ is nontrivial, can be obtained simply from degree considerations.

Theorem 6.1. *The homotopy orbit to* TR *spectral sequence*

$$E_1^{s,t}(\alpha) = V_*T[\alpha^{(n-s)}]_{hC_{p^s}} \Longrightarrow \mathrm{TR}^{n+1}_{\alpha+*}(A;p,V)$$

splits as a direct sum of the above spectral sequences, with no differentials between summands.

The differentials are determined by the following data. Let $e_j = u_j^{\epsilon_0}\lambda_1^{\epsilon_1}\lambda_2^{\epsilon 2}$ *and suppose*

$$x = v_c^i t^{d_j(\alpha)} e_{n-j}\mu_c^k[\alpha^{(j)}]$$

is a nontrivial class in the homotopy fixed point piece of $E_1^{n-j,*}$. *Let*

$$y_h = v_c^{i^{(h)}} t^{-p^{hc}k-\delta_c^h(\alpha^{(j-h+1)})} e_{n-j+h}[\alpha^{(j-h)}],$$

where

$$i^{(h)} = i - r(h-1)k - \sum_{0\le k\le h-2} \big[d_{j-h+1}(\alpha) - d_j(\alpha)\big]p^{ck}.$$

If x *survives to* $E_\rho^{n-j,*}$ *and the classes* $y_h \in V_*T[\alpha^{(j-h)}]^{tC_{p^{n-j+h}}}$ *are nonzero for* $1\le h\le \rho$ *then* $d_\rho(x) = \partial^h(y_\rho)$ *considered as a class in* $E_\rho^{n-j+\rho,*}$. *If at least one of the classes* y_h *for* $1\le h\le\rho$ *is zero then* $d_\rho(x) = 0$.

Proof. The class x in $V_*T[\alpha^{(j)}]_{hC_{p^{n-j}}}$ maps to a class with the same name in $\mathrm{TR}^{n-j+1}_{\alpha^{(j)}+*}(A;p,V)$. By comparing with the stable range we find that $\hat\Gamma_{n-j+1}(x) = y_1$. By construction of the spectral sequence this implies that $d_1(x) = \partial^h(y_1)$.

If $d_1(x) = 0$, then x lifts to a class x_1 in $\mathrm{TR}^{n-j+2}_{\alpha^{(j-1)}+*}(A;p,V)$. Let $z_1 = \Gamma_{n-j+1}(x_1)$ in $V_*T[\alpha^{(j-1)}]^{hC_{p^{n-j+1}}}$. While x_1, and hence z_1, may not be unique, we have a canonical choice for a representative for z_1 in the homotopy fixed point spectral sequence given by taking a representative for $\hat\Gamma_{n-j+1}(x)$ in the Tate spectral sequence and restricting to the second quadrant. We then have two cases.

Case 1: The class z_1 multiplies nontrivially by μ_c^N to the stable range. Because $V_*T[\alpha^{(j-1)}]^{hC_{p^{n-j+1}}}$ is isomorphic to $V_*T[\alpha^{(j-2)}]^{tC_{p^{n-j+2}}}$ in the stable range, this happens exactly when $y_2 = \hat\Gamma_{n-j+2}(x_1) \neq 0$. Again it follows by construction of the spectral sequence that $d_2(x) = \partial^h(y_2)$. The formula for $d_\rho(x)$ assuming $y_1,\dots,y_\rho$ are all nonzero follows by induction.

Case 2: The class z_1 multiplies trivially by μ_c^N to the stable range. In this case we find that $\hat\Gamma_{n-j+2}(x_2) = 0$, so $d_2(x) = 0$. By induction, x lifts to a class x_h in $\mathrm{TR}^{n-j+h+1}_{\alpha^{(j-h)}+*}(A;p,V)$ which multiplies trivially to the stable range for all h. Hence $d_\rho(x) = 0$ for all ρ. The same argument applies as soon as some y_h is zero. □

7. The $RO(S^1)$-graded TR–groups of $\mathbb{F}_p$

While Theorem 6.1 above tells us all the differentials in the spectral sequence converging to $\mathrm{TR}^{n+1}_{\alpha+*}(\mathbb{F}_p;p)$, we need some additional information to resolve the extension problems. As shown in [9], the extension problem is in fact quite delicate.

We observe that if we know the order of $\mathrm{TR}^{n+1}_{\alpha+*}(\mathbb{F}_p;p,\mathbb{Z}/p^l)$ for each $l\ge 1$, we can reconstruct $\mathrm{TR}^{n+1}_{\alpha+*}(\mathbb{F}_p;p)$. Let $T = T(\mathbb{F}_p)$. We find that

$$\pi_*(T[\alpha];\mathbb{Z}/p^l) \cong t^{d_0(\alpha)}E(\beta_l)\otimes P(\mu_0),$$

where β_l is in degree 1, and the Tate spectral sequence behaves as follows:

Lemma 7.1. *Consider the spectral sequence converging to $\pi_*(T[\alpha]^{tC_{p^n}};\mathbb{Z}/p^l)$. If $n < l$ there is a differential $d_{2n+1}(u_n) = tv_0^n$ and if $n \geq l$ there is a differential $d_{2l}(\beta_l) = v_0^l$.*

We can then record $\pi_*(T[\alpha]_{hC_{p^n}};\mathbb{Z}/p^l)$. As before, we split it into the Tate piece and the homotopy fixed point piece. If $n < l$ we find that the Tate piece is

$$\bigoplus_{k\geq n-d_0(\alpha)} E(\beta_l)\otimes P_n(v_0)\{t^{-k}[\alpha]\}[-1]$$
$$\bigoplus_{1\leq k+d_0(\alpha)\leq n-1} E(\beta_l)\otimes P_{k+d_0(\alpha)}(v_0)\{t^{-k}[\alpha]\}[-1]$$

Similarly, the homotopy fixed point piece is as follows:

$$\bigoplus_{k\geq n} E(\beta_l)\otimes P_{n+1}(v_0)\{t^{d_0(\alpha)}\mu_0^k[\alpha]\}$$
$$\bigoplus_{1\leq k\leq n} E(\beta_l)\otimes P_k(v_0)\{v_0^{n+1-k}t^{d_0(\alpha)}\mu_0^{k-1}[\alpha]\}$$

If $n \geq l$ we find that the Tate piece is

$$\bigoplus_{k\geq l-d_0(\alpha)} E(u_n)\otimes P_l(v_0)\{t^{-k}[\alpha]\}[-1]$$
$$\bigoplus_{1\leq k+d_0(\alpha)\leq l-1} E(u_n)\otimes P_{k+d_0(\alpha)}(v_0)\{t^{-k}[\alpha]\}[-1]$$

Similarly, the homotopy fixed point piece is as follows:

$$\bigoplus_{k\geq l} E(u_n)\otimes P_l(v_0)\{t^{d_0(\alpha)}\mu_0^k[\alpha]\}$$
$$\bigoplus_{1\leq k\leq l-1} E(u_n)\otimes P_k(v_0)\{v_0^{l-k}t^{d_0(\alpha)}\mu_0^k[\alpha]\}$$

Theorem 7.2. *Consider the spectral sequence*

$$E_1(\alpha) = \bigoplus_{0\leq s\leq n} \pi_*(T[\alpha^{(n-s)}]_{hC_{p^s}};\mathbb{Z}/p^l) \Longrightarrow \mathrm{TR}^{n+1}_{\alpha+*}(\mathbb{F}_p;p,\mathbb{Z}/p^l).$$

The differentials are determined by the following data. Suppose

$$x = p^i t^{d_j(\alpha)} u_{n-j}^{\epsilon}\mu_0^k[\alpha^{(j)}]$$

is a nontrivial class in the homotopy fixed point piece of $E_1^{n-j,*}$. Then $d_\rho(x)$ is given as in Theorem 6.1.

Now suppose

$$\bar{x} = p^i t^{d_j(\alpha)}\beta_l\mu_0^k[\alpha^{(j)}]$$

is a nontrivial class in the homotopy fixed point piece of $E_1^{n-j,*}$, and let

$$\bar{y}_h = \begin{cases} p^{i^{(h)}} t^{-k-\delta_0^h(\alpha^{(j-h+1)})}\beta_l[\alpha^{(j-h)}] & \text{if } n-j+h-l<0 \\ p^{i^{(h)}-(n-j+h-l)} t^{-k-1-\delta_0^h(\alpha^{(j-k+1)})} u_{n-j+h}[\alpha^{(j-h)}] & \text{if } n-j+h-l\geq 0 \end{cases}$$

If $\bar{x}$ survives to $E_\rho^{n-j,*}$ and the classes $\bar{y}_h \in \pi_*(T[\alpha^{(j-h)}]^{tC_{p^{n-j+h}}};\mathbb{Z}/p^l)$ are nonzero for $1 \leq h \leq \rho$ then $d_\rho(\bar{x}) = \partial^h(\bar{y}_\rho)$ considered as a class in $E_\rho^{n-j+\rho,*}$. If at least one of the classes $\bar{y}_h$ is zero then $d_\rho(\bar{x}) = 0$.

Proof. The proof is similar to the proof of Theorem 6.1. The extra factor of $p^{-(n-j+h-l)}$ comes from having $n-j+h-l$ homotopy orbit spectral sequences with a differential on β_l rather than a differential on some u_{j+h}. For each one, the possible differential, and possible successive lift of $\bar{x}$, behaves as if we had started with a multiple of $u_{n-j}\mu_0^{k+1}[\alpha^{(j)}]$ rather than a multiple of $\beta_l\mu_0^k[\alpha^{(j)}]$. □

We have written a Mathematica program which computes the $RO(S^1)$-graded TR groups of $\mathbb{F}_p$ using this algorithm.

8. The TR groups in degree $q-\lambda$

It is the TR–groups indexed by representations of the form $\alpha = q-\lambda$ that are most applicable to computations of algebraic K-theory. See, for example, Hesselholt and Madsen's computation of $K_q(\mathbb{F}_p[x]/(x^m),(x))$ in [11] and results of the authors and Hesselholt on $K_q(\mathbb{Z}[x]/(x^m),(x))$ in [2].

Proposition 8.1. *Consider the spectral sequence*

$$E_1^{s,t}(-\lambda) = \bigoplus_{0\le s\le n} V_*T[-\lambda^{(n-s)}]_{hC_{p^s}} \Longrightarrow \mathrm{TR}^{n+1}_{*-\lambda}(A;p,V)$$

for an actual representation λ. Then every nonzero class in the Tate piece is killed by a differential.

Proof. We prove this by induction, but with a slightly extended induction hypothesis. We consider a representation λ which is *almost* an actual representation, by which we mean that $d_i(\lambda) \ge d_{i+1}(\lambda)$ for $i \ge 1$ and $d_0(\lambda) \ge d_1(\lambda) - 1$.

Consider the first family of spectral sequences described in §6. It is enough to show that

$$z = t^{-p^{cn}k+\delta_c^n(\lambda')}[-1]$$

in the Tate piece of $E_1^{n,*}(-\lambda)$ is hit by a differential. For z to be nonzero we must have

$$p^{cn}k - d_0(\lambda) - \delta_c^n(\lambda') \ge 0.$$

Consider

$$x = t^{-d_1(\lambda)}\mu_c^{p^{c(n-1)}k-d_1(\lambda)-\delta_c^{n-1}(\lambda'')}$$

in the homotopy orbit piece of $E_1^{n-1,*}$. If

$$p^{c(n-1)}k - d_1(\lambda) - \delta_c^{n-1}(\lambda'') > r(n-2)$$

then x is nonzero and $d_1(x) = z$.

Now suppose

$$p^{c(n-1)}k - d_1(\lambda) - \delta_c^{n-1}(\lambda'') \le r(n-2).$$

Consider the class

$$y = v_c^{p^{c(n-1)}k-d_1(\lambda)-\delta_c^{n-1}(\lambda'')}t^{-p^{c(n-1)}k+\delta_c^{n-1}(\lambda'')}$$

in the Tate spectral sequence converging to $V_*T[-\lambda']^{tC_{p^{n-1}}}$. Then y is in filtration $2d_1(\lambda)$, which means that y is not in the first quadrant of the spectral sequence and hence $\partial^h(y) = 0$. Note that

$$0 \le p^{c(n-1)}k - d_1(\lambda) - \delta_c^{n-1}(\lambda'') \le r(n-2),$$

so y is nonzero in $V_*T[-\lambda']^{tC_{p^{n-1}}}$.

By assumption, $d_1(\lambda) \geq d_2(\lambda)$. Then we can consider a representation μ with $\mu'' = \lambda''$ and $d_1(\mu) = d_1(\lambda) - 1$. Then $\partial^h(y) \neq 0$ in $E_1^{n-1,*}(-\mu)$. By induction $\partial^h(y) = d_\rho(w)$ for some w in $E_\rho^{*,*}(-\mu)$. But then Theorem 6.1 implies that $d_{\rho+1}(w) = z$ in $E_{\rho+1}^{*,*}(-\lambda)$, proving the result.

The remaining two families of differentials can be treated in a similar way. □

We can now redo the calculation in [13]:

Corollary 8.2. *It follows that*

$$|TR^n_{q-\lambda}(\mathbb{F}_p;p)| = \begin{cases} p^n & \text{for } q = 2m \text{ and } d_0(\lambda) \leq m \\ p^{n-s} & \text{for } q = 2m \text{ and } d_s(\lambda) \leq m \leq d_{s-1}(\lambda) \\ 0 & \text{for } q \text{ odd} \end{cases}$$

Proof. In the case of $\mathbb{F}_p$, if we consider the spectral sequence

$$E_1^{s,*}(-\lambda) = \pi_* T[-\lambda^{(n-1-s)}]_{hC_{p^s}} \Longrightarrow \mathrm{TR}^n_{*-\lambda}(\mathbb{F}_p;p),$$

the only elements in odd total degree are those in the Tate piece. By Proposition 8.1, all those elements are killed, hence $|\mathrm{TR}^n_{q-\lambda}(\mathbb{F}_p;p)| = 0$ for q odd. In even degrees, since the differentials are surjective

$$|\mathrm{TR}^n_{2m-\lambda}(\mathbb{F}_p;p)| = \frac{\prod_s |E_1^{s,2m}|}{\prod_s |E_1^{s,2m-1}|} = \begin{cases} p^n & \text{for } d_0(\lambda) \leq m \\ p^{n-s} & \text{for } d_s(\lambda) \leq m \leq d_{s-1}(\lambda) \end{cases}$$

□

From the spectral sequence for $\mathrm{TR}^n_{q-\lambda}(\mathbb{F}_p;p,\mathbb{Z}/p)$ in §7 we conclude that the group $\mathrm{TR}^n_{q-\lambda}(\mathbb{F}_p;p)$ has just one summand. So we get the following result:

Theorem 8.3. *Let λ be a finite complex S^1-representation. Then*

$$\mathrm{TR}^n_{q-\lambda}(\mathbb{F}_p;p) \cong \begin{cases} \mathbb{Z}/p^n & \text{for } q = em \text{ and } d_0(\lambda) \leq m \\ \mathbb{Z}/p^{n-s} & \text{for } q = 2m \text{ and } d_s(\lambda) \leq m \leq d_{s-1}(\lambda) \\ 0 & \text{for } q \text{ odd} \end{cases}$$

This agrees with the result of Hesselholt and Madsen [13]. In the case of $A = \mathbb{Z}$ we can then prove Theorem 1.1:

Proof of Theorem 1.1. As described in §6 the E_1-term of the homotopy orbit to TR spectral sequence is composed of two families of small spectral sequences. In sufficiently high degrees we are left with the lower right-hand summands in the diagrams of §6. We first give the argument in high degrees and then describe the modifications needed in low degrees.

In the E_∞-term we are left with

$$E(\lambda_1) \otimes P_{r(n-1)+1}(v_1)\{t^{-d_0(\lambda)}\mu_1^{p^{n-1}k-d_0(\lambda)-\delta_1^{n-1}(\lambda')}\}$$

from the first family of spectral sequences, and

$$E(u_j) \otimes P_{r(j+1)}(v_1)\{t^{-d_0(\lambda)}\mu_1^{p^jk-d_0(\lambda)-\delta_1^j(\lambda')}\lambda_1\}$$

for $0 \leq j \leq n-2$ and k such that $v_p(k + d_j(\lambda) + \delta_1^{n-1-j}(\lambda^{(j+1)})) = 0$ from the second family. Assume $q = 2m$ is even. The length of $\mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p,\mathbb{Z}/p)$ is the number of different ways can $2m$ be written as

$$2m = 2d_0(\lambda) + 2p^nk - 2p(d_0(\lambda) + \delta_1^{n-1}(\lambda')) + a(2p-2)$$

for $0 \leq a \leq r(n-1)$ or

$$2m = 2d_0(\lambda) + 2p^{j+1}k - 2p(d_0(\lambda) + \delta_1^j(\lambda')) + (a+1)(2p-2)$$

for $0 \leq j \leq n-2$, $0 \leq a < r(j+1)$, and $v_p(k + d_j(\lambda) + \delta_1^{n-1-j}(\lambda^{(j+1)})) = 0$. Noting that

$$\delta_1^j(\lambda') = \delta_1^{n-1}(\lambda') - p^j d_j(\lambda) - p^j \delta_1^{n-1-j}(\lambda^{(j+1)})$$

we can rewrite these two equations as

$$2m - 2d_0(\lambda) + 2p(d_0(\lambda) + \delta_1^{n-1}(\lambda')) = 2p^n k + a(2p-2)$$

or

$$\begin{aligned} 2m - 2d_0(\lambda) &+ 2p(d_0(\lambda) + \delta_1^{n-1}(\lambda')) \\ &= 2p^{j+1}(k + d_j(\lambda) + \delta_1^{n-j-1}(\lambda^{(j+1)})) + (a+1)(2p-2) \end{aligned}$$

with the same conditions on a, j, and k as above. It follows that the length of $\mathrm{TR}^n_{2m-\lambda}(\mathbb{Z}; p, \mathbb{Z}/p)$ is the number of ways to write $b = m - d_0(\lambda) + p(d_0(\lambda) + \delta_1^{n-1}(\lambda'))$ as

$$b = p^n k + a(p-1)$$

where $0 \leq a \leq r(n-1)$ or

$$b = p^{j+1}k + a(p-1)$$

where $0 \leq j \leq n-2$, $1 \leq a \leq r(j+1)$, and $v_p(k) = 0$. Now, if $b = p^n k + a(p-1)$ with $1 \leq a \leq r(n-1)$ we can rewrite this as $b = p^{n-1}(pk) + a(p-1)$, and if $b = p^{j+1}k + a(p-1)$ with $1 \leq a \leq r(j)$ we can rewrite it as $b = p^j(pk) + a(p-1)$. Hence we have one class when $c = 0$ modulo p^n and one class for each way to write

$$b = p^{j+1}k + a(p-1)$$

with $0 \leq j \leq n-2$ and $r(j) < a \leq r(j+1)$, with no condition on $\nu_p(k)$. There is exactly one such pair (k, a) for each j, so we get $n-1$ classes, plus an additional class from the first family when $m = \delta_1^n(\lambda)$ modulo p^n corresponding to $a = 0$. The case $q = 2m+1$ odd is similar.

If $q \geq 2d_0(\lambda)$, but q is not sufficiently high that the spectral sequences degenerate with only the lower right hand summands in the E_∞ term, the result follows by comparing with $\pi_*(T[-\mu]^{tC_{p^n}}; \mathbb{Z}/p)$ for some μ with $\mu' = \lambda$. Using that the mod p homotopy groups of the Tate spectrum are $2p^n$-periodic and Tsalidis' Theorem [18], the result follows.

Part 2 and 3 follow by using that if $q < 2d_0(\lambda)$ we have an isomorphism

$$R : \mathrm{TR}^n_{q-\lambda}(\mathbb{Z}; p, \mathbb{Z}/p) \xrightarrow{\cong} \mathrm{TR}^{n-1}_{q-\lambda'}(\mathbb{Z}; p, \mathbb{Z}/p).$$

□

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